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NOTE

HOMOMORPHISMS AND AUTOMORPHISMS OF 2-D DE BRUIJN–GOOD GRAPHS

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In the paper all homomorphisms of a 2-D de Bruijn–Good graph with order $(m+1, n)$ over an arbitrary set to a 2-D de Bruijn–Good Graph with order (m, n) are given, the automorphism group of a 2-D de Bruijn–Good Graph is determined and it is shown that there are exactly six 2^n to 1 homomorphisms of a 2-D de Bruijn–Good Graph with order $(m+1, n)$ over the finite field $\mathbb{F}_2$ to a 2-D de Bruijn–Good Graph with order (m, n).

The 2-D de Bruijn–Good Graphs are related to 2-D de Bruijn Arrays (i.e. perfect maps) and Pseudo-Random Arrays. Recently 2-D de Bruijn Arrays and Pseudo-Random Arrays have received some attention, because these arrays can be used in two-dimensional range-finding, scrambling, various kinds of mask configurations in communication and coding. In this paper we give all homomorphisms of a 2-D de Bruijn–Good Graph with order $(m+1, n)$ over an arbitrary set M to a 2-D de Bruijn–Good Graph with order (m, n), determine all 2^n to one homomorphisms of such graphs over the finite field $\mathbb{F}_2$ and obtain the automorphism group of a 2-D de Bruijn–Good Graph over M.

Let m, n be positive integers. Let M be an arbitrary set. The de Bruijn–Good Graph over M with dimension 2 and order (m, n) is the digraph with vertex set V consisting of all $m \times n$ matrices over M, i.e.

$$V = \left\{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ & \cdots & & \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \,\middle|\, \alpha_{ij} \in M,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\right\},$$

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and with arc set the union of two sets E_1 and E_2 where

$$E_1 = \{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ & \cdots & & \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \to \begin{pmatrix} \alpha_{12} & \cdots & \alpha_{1n} & \alpha_{1n+1} \\ \alpha_{22} & \cdots & \alpha_{2n} & \alpha_{2n+1} \\ & & \cdots & \\ \alpha_{m2} & \cdots & \alpha_{mn} & \alpha_{mn+1} \end{pmatrix}$$

$$\mid \alpha_{ij} \in M,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n+1\},$$

$$E_2 = \{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ & \cdots & & \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \to \begin{pmatrix} \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ & \cdots & & \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \\ \alpha_{m+11} & \alpha_{m+12} & \cdots & \alpha_{m+1n} \end{pmatrix}$$

$$\mid \alpha_{ij} \in M,\ 1 \leqslant i \leqslant m+1,\ 1 \leqslant j \leqslant n\},$$

i.e. E_1 is the set of all arcs joining A_1 to A_2 where the last $n-1$ columns of A_1 are the first $n-1$ columns of A_2, E_2 is the set of all arcs joining B_1 to B_2 where the last $m-1$ rows of B_1 are the first $m-1$ rows of B_2. We shall use the terminology 'arcs of type E_i $(i=1,2)$'. This diagraph is denoted by $G_{m,n}(M)$ (or just $G_{m,n}$) and called a 2-dimensional or 2-D de Bruijn–Good Graph with order (m, n).

Let ϕ be a mapping from $G_{m+1,n}$ to $G_{m,n}$. ϕ is called a homomorphism of $G_{m+1,n}$ to $G_{m,n}$ if ϕ is a mapping from the vertex set of $G_{m+1,n}$ to the vertex set of $G_{m,n}$ that preserves arcs, i.e. for each arc $(A_1 \to A_2) \in E_i = E_i(G_{m+1,n})$ also $(\phi(A_1) \to \phi(A_2)) \in E_i = E_i(G_{m,n})$ with $i=1,2$. ϕ is called a 2^n to one homomorphism of $G_{m+1,n}(\mathbb{F}_2)$ to $G_{m,n}(\mathbb{F}_2)$, where $\mathbb{F}_2$ is the finite field with elements 0 and 1, if ϕ is a homomorphism of $G_{m+1,n}(\mathbb{F}_2)$ to $G_{m,n}(\mathbb{F}_2)$, mapping 2^n vertices of $G_{m+1,n}(\mathbb{F}_2)$ to one vertex of $G_{m,n}(\mathbb{F}_2)$.

Let ϕ be a mapping from $G_{m,n}(M)$ to itself. ϕ is called an automorphism of $G_{m,n}(M)$ if ϕ is a bijection on the vertex set V and arc set and preserves arcs, i.e. for each arc $(A_1 \to A_2) \in E_i = E_i(G_{m,n})$ also $(\phi(A_1) \to \phi(A_2)) \in E_i = E_i(G_{m,n})$ with $i=1,2$. Clearly all the automorphism of $G_{m,n}(M)$ form a group under composition. We denote it by $\mathrm{Aut}(G_{m,n}(M))$ (or just $\mathrm{Aut}(G_{m,n})$).

The main result of the paper is to determine all homomorphisms of $G_{m+1,n}(M)$ to $G_{m,n}(M)$, to show that there are exactly six 2^n to one homomorphisms of $G_{m+1,n}(\mathbb{F}_2)$ to $G_{m,n}(\mathbb{F}_2)$ and to prove $\mathrm{Aut}(G_{m,n}(M) \cong S_M$, where S_M is the symmetric group of the set M.

For convenience, sometimes we denote by $(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n)$ the matrix with columns $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$; and by $(\boldsymbol{b}_1\, \boldsymbol{b}_2 \cdots \boldsymbol{b}_m)^{\mathrm{T}}$ the matrix with rows $\boldsymbol{b}_1^{\mathrm{T}}, \boldsymbol{b}_2^{\mathrm{T}}, \ldots, \boldsymbol{b}_{\mathrm{m}}^{\mathrm{T}}$, where T denotes the transpose.

Let $A = (\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n)$ be a matrix of size $m+1$ by n over M, where $\boldsymbol{a}_j = (a_{1j}\, a_{2j} \cdots a_{m+1j})^{\mathrm{T}}$. By M^n, we shall mean the cartesian product of M with itself n times. Let ϕ be a homomorphism of $G_{m+1,n}(M)$ to $G_{m,n}(M)$. We denote the image of A under ϕ by

$$\phi(A) := (f_1(A) f_2(A) \cdots f_n(A)), \quad \text{i.e.}$$
$$\phi(A) := (f_1(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n) \cdots f_n(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n)),$$

where

$$f_1 : (M^{m+1})^n \rightarrow M^m.$$

If $(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n) \rightarrow (\boldsymbol{a}_2 \cdots \boldsymbol{a}_n\, \boldsymbol{x})$ is an arc of type E_1 then the image must also be an arc of type E_1, i.e. $\phi(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n) \rightarrow (\phi(\boldsymbol{a}_2 \cdots \boldsymbol{a}_n\, \boldsymbol{x}) \in E_1$. This implies that

$$f_j(\boldsymbol{a}_2 \cdots \boldsymbol{a}_n\, \boldsymbol{x}) = f_{j+1}(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n), \quad 1 \leqslant j \leqslant n. \tag{$*$}$$

Now $(*)$ implies that f_j does not depend on the last variable and f_{j+1} does not depend on the first variable. Since this is true for $1 \leqslant j < n$ we find by induction that in fact there is a function $f : M^{m+1} \rightarrow M^m$ such that for $1 \leqslant j \leqslant n$

$$f_j(\boldsymbol{a}_1\, \boldsymbol{a}_2 \cdots \boldsymbol{a}_n) = f(\boldsymbol{a}_j).$$

Applying the same argument to arcs of type E_2 (again induction) we find that there is a function $F : M^2 \rightarrow M$ such that

$$f(\boldsymbol{x}) = (F(x_1, x_2) F(x_2, x_3) \cdots F(x_m, x_{m+1}))^{\mathrm{T}},$$

where $\boldsymbol{x} = (x_1\, x_2 \cdots x_{m+1})^{\mathrm{T}}$. It is easy to check that each such function F indeed yields a homomorphism. If $M = \mathbb{F}_2$ and we require the homomorphism to be 2^n to 1, then F must take the value 0 and 1 two times. There are $\binom{4}{2} = 6$ such functions, namely:

$$F_1(x_1, x_2) = x_1, \quad F_2(x_1, x_2) = x_2, \quad F_3(x_1, x_2) = x_1 + x_2,$$
$$F_4(x_1, x_2) = x_1 + 1, \quad F_5(x_1, x_2) = x_2 + 1, \quad F_6(x_1, x_2) = x_1 + x_2 + 1. \tag{1}$$

Therefore we have the following theorems.

Theorem 1. *Let m, n be positive integers, M an arbitrary set and ϕ a mapping from $G_{m+1,n}(M)$ to $G_{m,n}(M)$. Then ϕ is a homomorphism from $G_{m+1,n}(M)$ to $G_{m,n}(M)$ iff there is a function $F : M^2 \rightarrow M$ such that $\phi((a_{ij})) = (F(a_{ij}, a_{i+1j}))$, i.e.*

$$\phi\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m+11} & a_{m+12} & \cdots & a_{m+1n} \end{pmatrix}$$
$$= \begin{pmatrix} F(a_{11}, a_{21}) & F(a_{12}, a_{22}) & \cdots & F(a_{1n}, a_{2n}) \\ F(a_{21}, a_{31}) & F(a_{22}, a_{32}) & \cdots & F(a_{2n}, a_{3n}) \\ & \cdots & & \\ F(a_{m1}, a_{m+11}) & F(a_{m2}, a_{m+12}) & \cdots & F(a_{mn}, a_{m+1n}) \end{pmatrix}$$

for all $A = (a_{ij}) \in V(G_{m+1,n}(M))$.

Theorem 2. *Let m, n be positive integers and $\mathbb{F}_2$ the finite field with elements* 0 *and* 1. *Then there are exactly six 2^n to 1 homomorphisms of $G_{m+1,n}(\mathbb{F}_2)$ to $G_{m,n}(\mathbb{F}_2)$. They are ϕ_i and $D\phi_i$, $i = 1, 2, 3$, where*

$$\phi_1\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m+11} & a_{m+12} & \cdots & a_{m+1n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$\phi_2\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m+11} & a_{m+12} & \cdots & a_{m+1n} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \\ a_{m+11} & a_{m+12} & \cdots & a_{m+1n} \end{pmatrix},$$

$$\phi_3\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m+11} & a_{m+12} & \cdots & a_{m+1n} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+a_{21} & a_{12}+a_{22} & \cdots & a_{1n}+a_{2n} \\ a_{21}+a_{31} & a_{22}+a_{32} & \cdots & a_{2n}+a_{3n} \\ & & & \\ a_{m1}+a_{m+11} & a_{m2}+a_{m+12} & \cdots & a_{mn}+a_{m+1n} \end{pmatrix},$$

and D is the dual automorphism of $G_{m,n}(\mathbb{F}_2)$

$$D\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+1 & a_{12}+1 & \cdots & a_{1n}+1 \\ a_{21}+1 & a_{22}+1 & \cdots & a_{2n}+1 \\ & \cdots & & \\ a_{m1}+1 & a_{m2}+1 & \cdots & a_{mn}+1 \end{pmatrix}.$$

In other words we get ϕ_1 by taking out the last row of the $(m+1)+n$ matrix, ϕ_2 by taking out the first row of the $(m+1)\times n$ matrix, ϕ_3 by adding each two adjacent rows of the $(m+1)\times n$ matrix.

Proof. By Theorem 1 and (1)

$$\phi_k((a_{ij})) = (F_k(a_{ij}, a_{i+1j})),$$
$$D\phi_k((a_{ij})) = (F_{3+k}(a_{ij}, a_{i+1j})), \quad k = 1, 2, 3. \quad \square$$

Theorem 3. *Let m, n be positive integers and M an arbitrary set. Then* $\mathrm{Aut}(G_{m,n}(M)) \cong S_M$, *where S_M is the symmetric group of the set M.*

Proof. Applying a similar argument as above to automorphisms of $G_{m,n}(M)$ we find that ϕ is an automorphism of $G_{m,n}(M)$ iff there is a permutation F of M such that $\phi((a_{ij})) = (F(a_{ij}))$. We can write $\phi = \phi_F$. The mapping $F \to \phi_F$ gives the isomorphism of the symmetric group S_M of M to the automorphism group $\mathrm{Aut}(G_{m,n}(M))$ of $G_{m,n}(M)$. $\square$

Notice that the automorphism group of $G_{m,n}(M)$ and homomorphisms of $G_{m+1,n}(M)$ to $G_{m,n}(M)$ do not depend on their order (or 'size'), but only depend on the set M.

For $n = 1$ there is an arc of type E_1 between any two vertices of $G_{m,n}(M)$. In this case we only consider the arc set E_2 and write $G_{m,1}(M) = G_m(M)$. Then $G_m(\mathbb{F}_2)$ is the well-known de Bruijn–Good Graph.

Corollary 1. *There are only two automorphisms of $G_m(\mathbb{F}_2)$: the identity mapping and the dual automorphism* (see [3]).

Corollary 2. *The six homomorphisms in Theorem 2 are exactly all homomorphisms of the de Bruijn–Good Graph $G_{m+1}(\mathbb{F}_2)$ to $G_m(\mathbb{F}_2)$ which are given in* [3].

Remark. The homomorphism ϕ_3 in Theorem 2 is a generalization of Lempel's homomorphism which is given in [1].

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